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Perturbative aspects of q-deformed dynamics

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Abstract. Within the framework of the q-deformed Heisenberg algebra a dynamical equation of q-deformed quantum mechanics is discussed. The perturbative aspects of the q-deformed Schrödinger equation are analyzed. General representations of the additional momentum-dependent interaction originating from the q-deformed effects are presented in two approaches. As examples, such additional interactions related to the harmonic-oscillator potential and the Morse potential are demonstrated.

Recently q-deformed quantum mechanics has attracted much attention as a possible modification of the ordinary quantum mechanics at short distances. According to present tests of quantum electrodynamics, quantum theories based on Heisenberg's commutation relation are correct at least down to 10^{-18} cm. The question arises whether there is a possible generalization of Heisenberg's commutation relation at shorter distances. In searching for such a possibility considerations of the space structure are a useful guide. If the space structure at such short distances exhibits a noncommutative property, and thus is governed by a quantum group symmetry, it has been shown that q-deformed quantum mechanics is a possible pre-quantum theory at short distances. In the literature different frameworks of q-deformed quantum mechanics were established [1–16].

The framework of the q-deformed Heisenberg algebra developed in [2,4] shows a clear physical content: its relation to the corresponding q-deformed boson commutation relations and the limiting process of the q-deformed harmonic oscillator to the undeformed one are clear. In this framework the q-deformed uncertainty relation shows an essential deviation from that of Heisenberg [14]: the ordinary minimal uncertainty relation is undercut. A nonperturbative feature of the q-deformed Schrödinger equation is that the energy spectrum exhibits an exponential structure [3,4,15]. The pattern of quark and lepton masses is qualitatively explained by such a q-deformed exponential spectrum [15].

In this paper we discuss *perturbative* aspects of the q-deformed Schrödinger equation in the above framework. The perturbative expansion of the q-deformed Hamiltonian possesses a complex structure, which amounts to some additional momentum-dependent interaction [2–4, 15]. There are two approaches to showing such q-deformed effects: One includes it in the kinetic-energy term, the other includes it in the potential. General results are presented, and as examples the harmonic-oscillator system and the Morse potential are discussed in some detail.

In the following, we first review the necessary background of q-deformed quantum mechanics. In terms of q-deformed phase space variables – the position operator X and the momentum operator P – the following q-deformed Heisenberg algebra has been developed [2,4]:

$$q^{1/2}XP - q^{-1/2}PX = \mathrm{i}U, \quad UX = q^{-1}XU, \quad UP = qPU, \tag{1}$$

where X and P are hermitian and U is unitary: $X^\dagger = X$, $P^\dagger = P$, $U^\dagger = U^{-1}$. Compared to the Heisenberg algebra the operator U is a new member, called the scaling operator. The necessity of introducing the operator U can be seen as follows.

The algebra (1) is based on the definition of the hermitian momentum operator P. However, if X is assumed to be a hermitian operator in a Hilbert space, the q-deformed derivative [4,17]

$$\partial_X X = 1 + qX\partial_X, \tag{2}$$

which embodies the noncommutativity of space, shows that the usual quantization rule $P \to -\mathrm{i}\partial_X$ does not yield a hermitian momentum operator. Reference [4] showed that a hermitian momentum operator P is related to ∂_X and X in a nonlinear way by introducing a scaling operator U:

$$\begin{aligned} U^{-1} &\equiv q^{1/2}[1 + (q-1)X\partial_X], \\ \bar{\partial}_X &\equiv -q^{-1/2}U\partial_X, \quad P \equiv -\frac{\mathrm{i}}{2}(\partial_X - \bar{\partial}_X), \end{aligned} \tag{3}$$

where $\bar{\partial}_X$ is the conjugate of ∂_X. The operator U is introduced in the definition of the hermitian momentum; thus, it closely relates to properties of the dynamics and plays an essential role in q-deformed quantum mechanics. The nontrivial properties of U imply that the algebra (1) has a richer structure than the Heisenberg commutation relation. In (1) the parameter q is a fixed real number. It is important to make distinctions for different realizations of the q-algebra by different ranges of q values [18–20]. Following [2,4] we only consider the case $q > 1$ in this paper. In the limit $q \to 1^+$ the scaling operator U reduces to the unit operator, and the algebra (1) reduces to the Heisenberg commutation relation.

The hermitian momentum P thus defined leads to q-deformation effects, which are exhibited by the dynamical equation. Equation (3) shows that the momentum P depends nonlinearly on X and ∂_X. Thus the q-deformed Schrödinger equation is difficult to treat. In this paper we demonstrate its perturbative aspects.

The q-deformed phase space variables X, P and the scaling operator U can be realized in terms of undeformed variables $\hat{x}$, $\hat{p}$ of the ordinary quantum mechanics, where $\hat{x}$, $\hat{p}$ satisfy: $[\hat{x},\hat{p}] = \mathrm{i}$, $\hat{x} = \hat{x}^\dagger$, $\hat{p} = \hat{p}^\dagger$. The variables X, P and the scaling operator U are related to $\hat{x}$, $\hat{p}$ by [4]:

$$X = \frac{\left[\hat{z} + \frac{1}{2}\right]}{\hat{z} + \frac{1}{2}}\hat{x}, \quad P = \hat{p}, \quad U = q^{\hat{z}}, \tag{4}$$

where $\hat{z} = -(\mathrm{i}/2)(\hat{x}\hat{p} + \hat{p}\hat{x})$ and $[A]$ is the q-deformation of A, defined by $[A] = (q^A - q^{-A})/(q - q^{-1})$. Using (4) it is easy to check that X, P and U satisfy (1).

From (4) it follows that X is represented as a function of $\hat{x}$ and $\hat{p}$ (note that $\hat{z} + (1/2) = -\mathrm{i}\hat{x}\hat{p}$):

$$X = \mathrm{i}(q - q^{-1})^{-1}(q^{(\hat{z}+1/2)} - q^{-(\hat{z}+1/2)})\hat{p}^{-1}. \tag{5}$$

Using (5) it is convenient to discuss the perturbative expansion of X. Let $q = \mathrm{e}^f = 1 + f$, with $0 < f \ll 1$. To the order f^2, X reduces to

$$X = \hat{x} + f^2 g(\hat{x},\hat{p}), \quad g(\hat{x},\hat{p}) = -\frac{1}{6}(1 + \hat{x}\hat{p}\hat{x}\hat{p})\hat{x}. \tag{6}$$

The q-deformed phase space $(X,\ P)$ governed by the q-algebra (1) is a q-deformation of the ordinary quantum mechanics phase space $(\hat{x},\ \hat{p})$; thus, the whole machinery of the ordinary quantum mechanics can be applied to the q-deformed quantum mechanics. By analogy, dynamical equations of the quantum system are the same for the undeformed phase space variables $\hat{x}$ and $\hat{p}$ and for the q-deformed phase space variables X and P. Thus the starting point for establishing perturbative calculations of the q-deformed Schrödinger equation is as follows: first one uses q-deformed phase space variables X and P to write down the Hamiltonian of the system, then one uses (4) to express X and P by the undeformed phase space variables $\hat{x}$ and $\hat{p}$.

The q-deformed Hamiltonian with potential $V(X)$ is

$$H(X,P) = \frac{1}{2\mu}P^2 + V(X). \tag{7}$$

For regular potentials $V(X)$, which are singularity free, to the order f^2 of the perturbative expansion, such potentials can be expressed by the undeformed variables $\hat{x}$ and $\hat{p}$ by

$$V(X) = V(\hat{x}) + \hat{H}_I^{(q)}(\hat{x},\hat{p}), \tag{8}$$

with the perturbation

$$\hat{H}_I^{(q)}(\hat{x},\hat{p}) = f^2 \sum_{k=1}^{\infty} \frac{V^{(k)}(0)}{k!} \left(\sum_{i=0}^{k-1} \hat{x}^{(k-1)-i} g(\hat{x},\hat{p}) \hat{x}^i \right), \tag{9}$$

where $V^{(k)}(0)$ is the kth derivative of $V(x)$ at $x = 0$ (x is the spectrum of $\hat{x}$). In (9) the ordering between the noncommutative quantities $\hat{x}$ and $g(\hat{x},\hat{p})$ is carefully considered. Substituting for $g(\hat{x},\hat{p})$ and summing over i, the above result can be expressed as

$$\hat{H}_I^{(q)}(\hat{x},\hat{p}) = \frac{f^2}{6} \sum_{k=1}^{\infty} \frac{V^{(k)}(0)}{k!} \hat{x}^k \times \left(k\hat{x}^2\partial_{\hat{x}}^2 + k(k+2)\hat{x}\partial_{\hat{x}} + \frac{1}{6}k(k-1)(2k+5) \right). \tag{10}$$

The remaining sum over k can be performed in terms of derivatives of the potential:

$$\hat{H}_I^{(q)}(\hat{x},\hat{p}) = \frac{f^2\hat{x}^2}{6} \left\{ \hat{x}V'(\hat{x})\partial_{\hat{x}}^2 + [\hat{x}V''(\hat{x}) + 3V'(\hat{x})]\partial_{\hat{x}} + \frac{1}{3}\hat{x}V'''(\hat{x}) + \frac{3}{2}V''(\hat{x}) \right\}. \tag{11}$$

For potentials with singular term X^{-k}, ($k = 1,2,3, \ldots$), one can use the following operator equation to treat the perturbation expansion:

$$\frac{1}{A+B} = \frac{1}{A} - \frac{1}{A}B\frac{1}{A} + \frac{1}{A}B\frac{1}{A}B\frac{1}{A} - \frac{1}{A}B\frac{1}{A}B\frac{1}{A}B\frac{1}{A} + \cdots, \tag{12}$$

where the norms of the operators A and B satisfy $\|B\| < \|A\|$. Thus to the order f^2 the perturbative expansion of $1/X$ reads

$$\frac{1}{X} = \frac{1}{\hat{x}} - f^2\frac{1}{\hat{x}}g(\hat{x},\hat{p})\frac{1}{\hat{x}}. \tag{13}$$

For the energy shift, in the state $|n\rangle$, corresponding to (11), we may integrate by parts, and obtain

$$\Delta\hat{E}_n^{(q)} = -\frac{f^2}{36}\int_{-\infty}^{\infty} \mathrm{d}x \left\{ \psi_n^{(0)*}(x)V(x) \times \left[2x^3\partial_x^3 + 9x^2\partial_x^2 - 3\right] \psi_n^{(0)}(x) + \mathrm{h.c.} \right\}, \tag{14}$$

where $\psi_n^{(0)}$ is the unperturbed wave function. One can use the Schrödinger equation and rewrite this as

$$\Delta\hat{E}_n^{(q)} = \frac{f^2}{6}\int_{-\infty}^{\infty} \mathrm{d}x \psi_n^{(0)*}(x) \left(V(x)\{1 - 4\mu x^2[V(x) - E]\} - \frac{2}{3}\mu E x^3 V'(x) \right) \psi_n^{(0)}(x), \tag{15}$$

where E is the unperturbed energy.

There is another set of variables $\tilde{x}$ and $\tilde{p}$ of an undeformed algebra, which are obtained by a canonical transformation of $\hat{x}$ and $\hat{p}$ [4]:

$$\tilde{x} = \hat{x}F^{-1}(\hat{z}), \quad \tilde{p} = F(\hat{z})\hat{p}, \tag{16}$$

where (note that $\hat{z} - (1/2) = -\mathrm{i}\hat{p}\hat{x}$)

$$F^{-1}(\hat{z}) = \frac{\left[\hat{z} - \frac{1}{2}\right]}{\hat{z} - \frac{1}{2}} \,. \tag{17}$$

The variables $\tilde{x}$ and $\tilde{p}$ thus defined also satisfy the undeformed algebra: $[\tilde{x}, \tilde{p}] = \mathrm{i}$, and $\tilde{x} = \tilde{x}^\dagger$, $\tilde{p} = \tilde{p}^\dagger$. Thus $\tilde{p} = -\mathrm{i}\partial_{\tilde{x}}$. The q-deformed variables X, P and the scaling operator U are related to $\tilde{x}$ and $\tilde{p}$ as follows:

$$X = \tilde{x}, \quad P = F^{-1}(\tilde{z})\tilde{p}, \quad U = q^{\tilde{z}}, \tag{18}$$

where $\tilde{z} = -(\mathrm{i}/2)(\tilde{x}\tilde{p} + \tilde{p}\tilde{x})$; and with $F^{-1}(\tilde{z})$ defined by (17) for the variables $(\tilde{x}, \tilde{p})$. From (16)–(18) it follows that X, P and U also satisfy (1), and (18) is equivalent to (4).

Using (18) to the order f^2 the perturbative expansions of P and the kinetic energy $P^2/(2\mu)$ read

$$P = \tilde{p} + f^2 h(\tilde{x}, \tilde{p}), \quad h(\tilde{x}, \tilde{p}) = -\frac{1}{6}(1 + \tilde{p}\tilde{x}\tilde{p}\tilde{x})\tilde{p}, \tag{19}$$

and

$$\frac{1}{2\mu}P^2 = \frac{1}{2\mu}\tilde{p}^2 + \tilde{H}_I^{(q)}(\tilde{x}, \tilde{p}), \tag{20}$$

with

$$\begin{aligned}\tilde{H}_I^{(q)}(\tilde{x}, \tilde{p}) &= \frac{1}{2\mu}f^2[\tilde{p}h(\tilde{x}, \tilde{p}) + h(\tilde{x}, \tilde{p})\tilde{p}] \\ &= -\frac{1}{12\mu}f^2[2\tilde{x}^2\partial_{\tilde{x}}^4 + 8\tilde{x}\partial_{\tilde{x}}^3 + 3\partial_{\tilde{x}}^2]. \end{aligned} \tag{21}$$

Equations (20) and (21) show that in the $(\tilde{x}, \tilde{p})$ system the perturbative contribution comes from the kinetic-energy term.

Similar to (14) and (15) (using the Schrödinger equation and integrating by parts), one can write the energy shift corresponding to (21) as

$$\begin{aligned}\Delta\tilde{E}_n^{(q)} = \frac{f^2}{6}\int_{-\infty}^{\infty} &\mathrm{d}x\psi_n^{(0)*}(x)[V(x) - E] \\ &\times\{1 - 4\mu x^2[V(x) - E]\}\psi_n^{(0)}(x). \end{aligned} \tag{22}$$

The two expressions for the energy shift, (15) and (22), are in fact equal, since the difference is given by

$$\begin{aligned}\frac{f^2}{6}E\int_{-\infty}^{\infty}\mathrm{d}x\psi_n^{(0)*}(x)\Bigg\{1 - 4\mu x^2[V(x) - E] \\ -\frac{2}{3}x^3\mu V'(x)\Bigg\}\psi_n^{(0)}(x) = 0. \end{aligned} \tag{23}$$

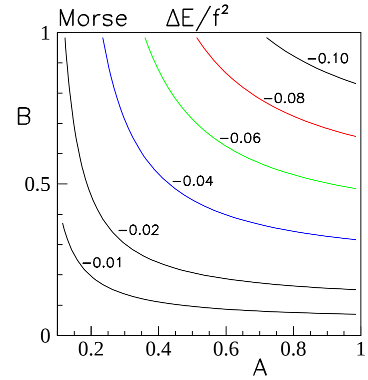

Fig. 1. Energy shift for the Morse potential, $\Delta E/f^2$, versus A and B, for $\alpha = 1$

From this last form, (22), it is easy to see that the energy shift is negative since $\langle n|V|n\rangle < E$. Thus,

$$\Delta E_n^{(q)} < 0. \tag{24}$$

As a first application we consider the q-deformed "harmonic" system described by the Hamiltonian

$$H(X, P) = \frac{1}{2\mu}P^2 + \frac{1}{2}\mu\omega^2 X^2 \,. \tag{25}$$

First we calculate $\Delta\tilde{E}_n^{(q)}$ in the $(\tilde{x}, \tilde{p})$ system. From (21) or (22) it follows that the shifts of the energy levels are

$$\Delta\tilde{E}_n^{(q)} = -\frac{f^2\omega}{48}\left(4n^3 + 6n^2 + 20n + 9\right). \tag{26}$$

In the $(\hat{x}, \hat{p})$ system the only nonzero term in (9) is $V^{(2)}(0) = \mu\omega^2$; thus, (9) reduces to

$$\begin{aligned}&\hat{H}_I^{(q)}(\hat{x}, \hat{p}) \\ &= -\frac{1}{12}f^2\mu\omega^2[\hat{x}(1 + \hat{x}\hat{p}\hat{x}\hat{p})\hat{x} + (1 + \hat{x}\hat{p}\hat{x}\hat{p})\hat{x}^2] \\ &= \frac{1}{12}f^2\mu\omega^2[2\hat{x}^4\partial_{\hat{x}}^2 + 8\hat{x}^3\partial_{\hat{x}} + 3\hat{x}^2]. \end{aligned} \tag{27}$$

The corresponding energy shift, which can also be obtained from (15), is easily seen to be identical to that of (26).

As noted above, the shift in (26) is negative, and it increases with n, leading eventually to a breakdown of perturbation theory for $n \sim (12/f^2)^{1/3}$. The tendency exhibited by (26) agrees with the observation that for the q-deformed harmonic oscillator the spectrum has an upper bound [5].

In the limiting case $q \to 1^+$ we have $H(X, P) \to H_{\mathrm{un}}(\hat{x}, \hat{p}) = (1/(2\mu))\hat{p}^2 + (1/2)\mu\omega^2\hat{x}^2$. Only in this sense $H(X, P)$ defined in (25) is called the q-deformed "harmonic" system.

As another example, we study the Morse potential [21] in its "supersymmetric" form [22], where the ground state energy vanishes. It is given by the potential

$$V(x) = A^2 + B^2\mathrm{e}^{-2\alpha x} - 2B\left(A + \frac{\alpha}{2\sqrt{2\mu}}\right)\mathrm{e}^{-\alpha x}. \tag{28}$$

The corresponding energy shift can be obtained from either (15) or (22); the result is shown in Fig. 1 for $\alpha = 1$, $\mu = 1$, and some range of A and B. For the harmonic oscillator, we saw that the shift increased in magnitude with the unperturbed energy. This is not the case for the Morse potential, where the shift may increase or decrease with the unperturbed energy, depending on the parameters.

It should be emphasized again that $\tilde{H}_I^{(q)}(\tilde{x}, \tilde{p})$ originates from the kinetic term, whereas $\hat{H}_I^{(q)}(\hat{x}, \hat{p})$ originates from the potential. At the level of operators, these two Hamiltonians are different. However, they differ only by a quantity whose expectation value vanishes.

At short distances, where q-deformation might be relevant, one also expects quantum mechanics to break down and to have to be replaced by some kind of field theory. Some progress is being made in this area [23]. In a more realistic theory along such lines, some features of q-deformed quantum mechanics may survive. It is therefore hoped that studies of q-deformed dynamics at the level of quantum mechanics will give some clue for the further development.

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